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FIXED POINT THEOREM IN ORDERED METRIC SPACES

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ABSTRACT

The main intention of this paper is to extend the result of Piri and Kunam [1] on a complete ordered metric space.

KEYWORDS: Fixed Point, F- Suzuki Contraction, Ordered Metric Space

INTRODUCTION

One of the most important result of the functional analysis is the Banach Contraction mapping. Various other generalization can be found out in the recent years. In the year 2012, Wardowski [4] have introduced a new type of contractions called *F*- contraction. Recently in the year 2014 Piri, H., Kumam, P [1] have extended the result of Wardowski [4] by considering the class of function satisfying different properties. In the same paper Piri, H., Kumam, P. [1] have defined the new type of contraction known as *F*-Suzuki contraction. Then in the year 2015, Karapinar, E., A.K,Marwan., Piri, H., O'Regan, D., [2] have extended the new version using the *F* Suzuki contraction of Piri, H., Kumam, P. [1]. Fixed points for the ordered metric spaces using *F*contraction have been obtained by Cosentino and Vetro [5] in the year 2014. The present paper focuses on bringing the extended versions of *F*-Suzuki contraction on ordered metric spaces.

Definition 1: [1] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an *F*-Suzuki contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2} d(x,Tx) < d(x,y) \Rightarrow \tau + F(d(Tx,Ty)) < F(d(x,y)),$$

Where F satisfies following conditions:

(F1) *F* is strictly increasing, *i.e.* for all $x, y \in R_+$ such that $x < y, F(x) < F(y)$;

(F2) inf $F = -\infty$.

(F3) *F* is continuous on $(0,\infty)$.

Let $\Im$ denote the class of functions satisfying (F1),(F2) and(F3).

Theorem 1: [1] Let (X,d) be a complete metric space and $T : X \to X$ be an *F*-Suzuki contraction. Then *T* has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

The extended form of the above result can also be found in the paper [2] where the authors have defined the conditionally *F*-contraction of type (A) on a metric-like space.

Secelean [3] have proved the following lemma

Lemma 1: [3] Let $F : R_+ \to R$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:

- If $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} (\alpha_n) = 0$.
- If $\inf F = -\infty$ and $\lim_{n\to\infty} (\alpha_n) = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Definition 2: [2] Let (X, d) be a metric-like space. A mapping $T : X \to X$ is said to be conditionally *F*-contraction of type (A) if there exists $F \in \Im$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \tau + F(d(Tx,Ty)) < F(M_T(x,y)),$$

Where
$$M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{4}\right\}$$

Theorem 2: [2] Let (X, d) be a metric-like space. If T is aconditionally *F*-contraction of type (A), then *T* has a fixed point $x^* \in X$

The present paper extends the above result in an ordered metric spaces.

Definition 3: Let $(X, \leq)$ be apartially ordered set and $T : X \to X$ a mapping. We say that *T* is non decreasing if for $x, y \in X$

$$x \leq y \Rightarrow Tx \leq Ty.$$

MAIN RESULTS

Theorem 3: Let $(X, \leq)$ be apartially ordered set and suppose that there exist a metric *d* in *X* such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that if there exists $F \in \Im$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x,T^p x) < d(x,y) \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1}$$

Where $p \in N$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ then *T* has a fixed point $x^* \in X$.

Proof: Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way

If $x_0 = Tx_0$ then there is nothing to prove. So $x_0 < Tx_0$ and T is non decreasing mapping we obtain

$$x_0 < Tx_0 \le T^2 x_0 \le T^3 x_0 \le T^4 x_0 \le .... \le T^n x_0 \le T^{n+1} x_0 \le .... \quad (1)$$

Put $x_{n+1} = Tx_n = T(T^n x_0) = T^{n+1} x_0$. Consequently the sequence $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.We have $x_n \le x_{n+1} = Tx_n$, if $x_n = Tx_n$ then we have a fixed point so we assume that $x_n < Tx_n$ i.e., $d(x_n, Tx_n) > 0$. Now since T is non-decreasing mapping $x_n < T^p x_n$ for any $p, n \in N$.

So we have

$$\frac{1}{2} d(x_n, T^p x_n) < d(x_n, T^p x_n).$$

So from (1) we have,

$$\tau + F(d(Tx_n, T(T^p x_n))) \le F(d(x_n, T^p x_n)).$$

$$\Rightarrow F(d(Tx_n, T(T^p x_n))) \le F(d(x_n, T^p x_n)) - \tau. \quad (2)$$

$$\Rightarrow F(d(x_{n+1}, x_{n+p+1}))) \le F(d(x_n, x_{n+p})) - \tau.$$

$$\le F(d(x_{n-1}, x_{n+p-1}))) - 2\tau.$$

$$\le F(d(x_{n-2}, x_{n+p-2}))) - 3\tau.$$

$$\le F(d(x_{n-3}, x_{n+p-3}))) - 4\tau.$$

$$\le F(d(x_0, x_p))) - (n+1)\tau.$$

So we have

$$F(d(Tx_n, T(T^p x_n))) \le F(d(x_0, x_p)) - (n+1)\tau.$$

$$F(d(x_{n+1}, x_{n+1+p}))) \le F(d(x_0, x_p)) - (n+1)\tau.$$

$$F(d(x_m, x_{m+p}))) \le F(d(x_0, x_p)) - m\tau.$$

$$\lim_{m\to\infty} F(d(x_m, x_{m+p}))) = -\infty. \quad (3)$$

Therefore from lemma 1 we get,

$$\lim_{m\to\infty} d\left(x_m, x_{m+p}\right) = 0 \quad (4)$$

Which shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence Since X is a complete metric space, there exists $x^* \in X$ such that

$$\lim_{m\to\infty} d\left(x_n, x^*\right) = 0 \quad (5)$$

Since, $x_{n+1} < x_{n+p}$, $\Rightarrow d(x_{n+p}, x^*) < d(x_{n+1}, x^*) \Rightarrow d(T^p x_n, x^*) < d(Tx_n, x^*)$. (6)

Now we claim that

$$\frac{1}{2} d(x_n, T^p x_n) < d(x_n, x^*) \text{ or, } \frac{1}{2} d(Tx_n, T^p(Tx_n)) < d(Tx_n, x^*), \quad n \in N \quad (7)$$

If then there exists some $m \in N$ such that

$$\frac{1}{2} d(x_n, T^p x_n) \geq d(x_n, x^*) \text{ and } \frac{1}{2} d(Tx_n, T^p(Tx_n)) \geq d(Tx_n, x^*). \quad (8)$$

$$\Rightarrow 2d(x_n, x^*) \leq d(x_n, T^p x_n) \leq d(x_n, x^*) + d(x^*, T^p x_n) < d(x_n, x^*) + d(x^*, Tx_n).$$

$$\Rightarrow d(x^*, x_n) < d(x^*, Tx_n). \quad (9)$$

Now from (7) and (8) we get,

$$\Rightarrow d(x^*, x_n) < d(x^*, Tx_n) \leq \frac{1}{2} d(Tx_n, T^p(Tx_n)). \quad (10)$$

Again from (2) since $\tau > 0$, we get,

$$F\left(d\left(Tx_n, T(T^p x_n)\right)\right) < F\left(d\left(x_n, T^p x_n\right)\right).$$

Also from (F1),

$$d\left(Tx_n, T(T^p x_n)\right) < d\left(x_n, T^p x_n\right). \quad (11)$$

Therefore it follows from (11), (10), (8), and (6).

$$d\left(Tx_n, T(T^p x_n)\right) < d\left(x_n, T^p x_n\right).$$

$$< d(x_n, x^*)) + d\left(x^*, T^p x_n\right).$$

$$< \frac{1}{2} d(Tx_n, T^p(Tx_n)) + d(x*, Tx_n).$$

$$< \frac{1}{2} d(Tx_n, T^p(Tx_n)) + \frac{1}{2} d(Tx_n, T^p(Tx_n)).$$

$$< d(Tx_n, T^p(Tx_n)).$$

Which is a contradiction Hence (7) holds so from first part of (7) we have

$$\tau + F(d(Tx_n, Tx*)) \leq F(d(x_n, x*)).$$

Then by continuity of $F$and using (5), and (F2)we get,

$$\lim_{n\to\infty} F(d(Tx_n, Tx*)) < F(\lim_{n\to\infty} d(x_n, x*)).$$

$$\lim_{n\to\infty} F(d(Tx_n, Tx*)) < F(0) = \inf F.$$

$$\lim_{n\to\infty} F(d(Tx_n, Tx*)) = -\infty.$$

Therefore from lemma 1 we get,

$$\lim_{n\to\infty} d(Tx_n, Tx*) = 0.$$

Hence, we have

$$d(x*, Tx*) = \lim_{n\to\infty} d(x_{n+1}, Tx*) = \lim_{n\to\infty} d(Tx_n, Tx*) = 0.$$

Which shows that $x*$ is the fixed point of T.

From second part of (7) we have,

$$\tau + F(d(T^2 x_n, Tx*)) \leq F(d(Tx_n, x*)) = F(d(x_{n+1}, x*))$$

Similarly, we get

$$\lim_{n\to\infty} F(d(T^2 x_n, Tx*)) = -\infty.$$

Using lemma 1 we obtain

$$\lim_{n\to\infty} d(T^2 x_n, Tx*) = 0.$$

Therefore

$$d(x*, Tx*) = \lim_{n\to\infty} d(x_{n+2}, Tx*) = \lim_{n\to\infty} d\left(T^2 x_n, Tx*\right) = 0$$.

Which shows that $x*$ is the fixed point of T.

Uniqueness: we now show that the fixed point is unique. If suppose that there are two distinct fixed points of *T*, $x*$ and $y*$ .that is $Tx* = x* \neq y* = Ty*$, then $d(x*, y*) > 0$. So, we have

$$\frac{1}{2} d(x*, T^p x*) < d(x*, y*)$$.

Then for the assumption of the theorem , we obtain

$$F(d(x*, y*)) = F\left(d\left(Tx*, T^p y*\right)\right) < \tau + F\left(d\left(Tx*, T^p y*\right)\right) \leq F(d(x*, y*))$$.

Which is a contradiction

Remark: If $p = 1$ then we get the version of *F*-suzuki contraction as presented by Piri, H., Kumam, P [1].

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